Minimum gain identifiable when pseudorandom binary sequences are used for system identification in noisy conditions

J.N. Davidson, D.A. Stone and M.P. Foster

This letter presents a method to determine the minimum gain of the system which can be identified by pseudorandom binary sequence (PRBS)-based approaches when noise is present in the output signal. A formula for the minimum gain is derived in terms of required noise resilience from the discrete Fourier transforms of PRBS signals and white noise. The formula allows an engineer to discount noise-affected low-gain portions of the transfer function. The calculated minimum gain is compared graphically to noisy bode plots and good agreement is shown with the actual noise level, demonstrating the usefulness of the result.

Introduction: Pseudorandom binary sequences (PRBS) are a special class of signals which are produced by linear shift feedback registers and whose power spectral density is approximately flat over the frequencies of interest. They are exploited in system identification to efficiently identify the transfer function of a system by applying a PRBS signal to its input and sampling its output. Applications of system identification by PRBS are broad and include modelling of batteries [1] and thermal systems [2], in addition to many other uses.

The transfer function of a system, H, is most simply identified by taking the quotient of the Fourier transforms of its output and input when excited by PRBS [3]. In practical systems, noise will be present in the measured output voltage and this is especially problematic in systems with low-gain transfer functions where the input signal is of limited amplitude, as the signal-to-noise ratio at frequencies of interest is insufficient for accurate system identification. One example of this is thermal systems whose gains fall off rapidly at modest frequencies.

Previous literature has reported methods of reducing noise; for example, using cross-correlation [4]. In this letter, however, we calculate the minimum gain identifiable in a system with additive white Gaussian noise (AWGN) present in its output, allowing an engineer to discount noise-affected frequencies in the identified response and determine whether noise-reduction techniques are required.

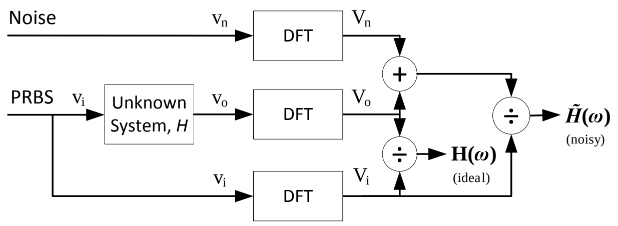

Fig. 1 *Process for system identification using PRBS showing noiseless output, $H(\omega)$, and noisy output, $\tilde{H}(\omega)$.*

Theoretical: For an unknown system, H, with AWGN present in the output, the experimental process to identify H can be modelled as in Fig. 1. In a noiseless system, $H(\omega)$ can be exactly identified over the useful frequency band of the PRBS. However, when noise, v_n, is present, only an approximation to $H(\omega)$ is found, which we denote $\tilde{H}(\omega)$. Although in reality the noise is present before the DFT is taken, it is shown being added afterwards in Fig. 1. The linearity of the DFT means this has no effect on the result, but the mathematics is simplified. The transfer function calculated is therefore the sum of the actual transfer function plus a noise-dependent term. This is shown for angular frequency, ω, in (1).

$$\tilde{H}(\omega) = \frac{V_o(\omega) + V_n(\omega)}{V_i(\omega)} = H(\omega) + \frac{V_n(\omega)}{V_i(\omega)} \tag{1}$$

where V_i, V_n and V_o are the DFTs of the input, noise and ideal output signals respectively. The power spectral density (PSD) of a PRBS can be shown [5, 6] to be (2). Over the useful frequency range, the spectrum is approximately flat with height derived from (2) when $f \to 0$.

$$\Phi_{xx}(f) = \frac{V^2}{f_P}\frac{N+1}{N}\left[\frac{\sin(f\pi/f_P)}{f\pi/f_P}\right]^2 \tag{2}$$

where $\Phi_{xx}(f)$ is the PSD at frequency f of a bipolar PRBS, amplitude V, length $N = 2^n - 1$, where n is the shift-register length, clocked at frequency f_P. To calculate the DFT, we note that the PSD of a PRBS is the Fourier transform of its autocorrelation function [6] and also that the Fourier transform of its autocorrelation function is the squared modulus of its Fourier transform [7]. Converting the Fourier transform to the DFT (and scaling accordingly), the PSD can therefore be written

$$\Phi_{xx}[k] = \frac{|V_i|^2}{n_s f_s} \tag{3}$$

where V_i is the DFT of the PRBS, f_s is the sampling rate, n_s is the total number of samples and k is the discrete frequency index where $f = kf_P/N$. The $n_s f_s$ divisor accounts for the scaling difference between the continuous Fourier transform and the DFT. Substituting (2) into (3) when $f \to 0$ (and noting that sequence duration $N/f_P = n_s/f_s$ for a PRBS) gives an approximation of the DFT of a PRBS over its useful range, given in (4).

$$|V_i| \cong V\frac{f_s}{f_P}\sqrt{N+1} \tag{4}$$

The DFT of AWGN, V_n, consists of real and imaginary components which are independently and normally distributed [8]. $|V_n|$ is therefore Rayleigh distributed [9]. From Parseval's relation [7] it is possible to state the following.

$$\frac{1}{n_s}\sum_{i=0}^{n_s-1}|V_n[i]|^2 = n_s\overline{v_n^2} \tag{5}$$

where $\overline{v_n^2}$ is the noise power, a measurement of the noise present in the signal. $|V_n|$ takes the form of a mean value with random perturbations. The mean value is the mean of the corresponding Rayleigh distribution and the amplitude of the random perturbations is a function of the standard deviation. The mean and standard deviation are derived in literature [9] and are given in (6) and (7) respectively. Using (5), they are also expressed in terms of noise power, $\overline{v_n^2}$.

$$\overline{|V_n|} = \sqrt{\frac{\pi}{2}\frac{1}{2n_s}\sum_{i=0}^{n_s-1}|V_n[i]|^2} = \sqrt{\frac{\pi}{4}n_s\overline{v_n^2}} \tag{6}$$

$$\text{stdev}\,|V_n| = \sqrt{\frac{4-\pi}{2}\frac{1}{2n_s}\sum_{i=0}^{n_s-1}|V_n[i]|^2} = \sqrt{\left(1-\frac{\pi}{4}\right)n_s\overline{v_n^2}} \tag{7}$$

$|V_n|$ consists of these two components: the mean value and random perturbations which are in effect bounded by a multiple of the standard deviation. From (1) and (4), the minimum gain identifiable in the presence of noise, H_{min}, can be found when

$$|H(\omega)| \approx \frac{\max|V_n|}{V\frac{f_s}{f_P}\sqrt{N+1}} \tag{8}$$

The numerator is a combination of the mean of the noise DFT plus a multiple, Δ, of the standard deviation. To calculate H_{min}, (6) and (7) are substituted into (8) noting that the number of samples, n_s is Nf_s/f_P. This value is given in (9)

$$H_{\text{min}} = \frac{1}{2}\sqrt{\frac{\overline{v_n^2}}{V^2}\frac{f_P}{f_s}\frac{N}{N+1}}\left(\sqrt{\pi} + \Delta\sqrt{4-\pi}\right) \tag{9}$$

where Δ is the number of standard deviations of noise perturbations which is acceptable for the application. We will refer to H_{min} at $\Delta = 0$ as the mean noise level because it is the gain in a bode plot of $\tilde{H}$ around which the noise is centred.

Results: To illustrate the effect of noise, a 1 V 8-bit ($N = 255$) bipolar PRBS is applied to the system, H, in (10) under varying noise power conditions. This system is a three element 1 kΩ–1 µF RC ladder. The PRBS clock frequency is 30 kHz and the output is sampled synchronously at 30 kHz.

$$H(s) = \frac{1}{(10^{-9})s^3 + (6 \times 10^{-6})s^2 + (5 \times 10^{-3})s + 1} \qquad (10)$$

where s is the Laplace operator variable. Fig. 2 shows the relationship between the noise power, $\overline{v_n^2}$, and the bode plot of the identified transfer function, $\tilde{H}(\omega)$. $\tilde{H}(\omega)$ is presented for several mean noise levels calculated from (9) at $\Delta = 0$, each of which is indicated by a dashed line. There is good agreement between the mean noise level and the gain at which noise dominates the identified transfer function. This confirms that for AWGN the gain where noise dominates is calculable from the Rayleigh distribution.

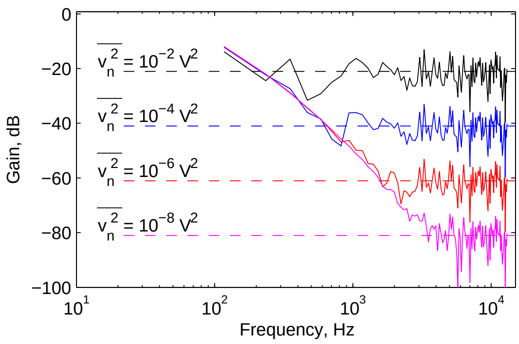

Fig. 2 *Mean noise level at varying noise power*

——— Gain identified in presence of noise, $|\tilde{H}(\omega)|$

- - - - - - Corresponding mean noise level, H_{min} at $\Delta = 0$

Fig. 3 shows the relationship between the number of standard deviations of noise, Δ, above the mean which may be discarded for an acceptably noise-free result. As Δ increases, the noise remaining in the signal decreases but so too does the useful bandwidth, thus a compromise is required. The level of noise acceptable is dependent on the application but, by inspection, truncating the signal when $|H|$ falls under the $\Delta = 2$ line removes the majority of the noise while leaving a large part of the transfer function intact.

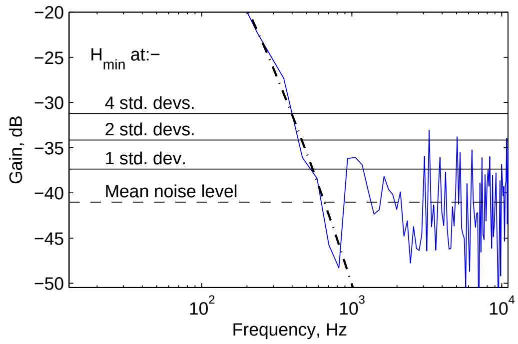

Fig. 3 *Minimum gain identifiable at varying standard deviations relative to the mean noise level*

——— Gain identified in presence of noise, $|\tilde{H}(\omega)|$

– · – · – Ideal gain, $|H|$

It is therefore possible to predict the minimum gain of a system which can be identified using PRBS techniques. With this information, an engineer can recognise when the transfer function has become unacceptably affected by noise and hence confidently discard the affected portion of the result.

Conclusion: In the frequency domain, additive white Gaussian noise has a Rayleigh-distributed DFT with non-zero mean. Using this property, the minimum gain of the system identifiable by PRBS in a noisy environment has been calculated. The relationship between the mean noise level and the identified transfer function has been presented with good agreement shown between the predicted and the actual gain at which noise dominates. An acceptable noise level is discussed in terms of the number of standard deviations above the mean noise level in a signal. By choosing a sensible number of standard deviations of noise acceptable, an engineer can determine the gain under which the effect of noise is too great and hence discard the affected portions of the identified transfer function.

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