Self-Organisation in the Van der Pol generator

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Introduction

Self-organised criticality is a model applicable to many real world systems such as climate, stock markets or forest fires (Zaripov 2008), (Klimontovich 1995). In this particular project, I will eventually be looking at using it to model sustainable energy generation, using a generalised concept of entropy.

For this first half of the project I concentrated on investigating the properties of a typical self-organising system: the Van der Pol oscillator. The modelling was done in MatLab, and the three environments imposed on the oscillator were:

1. Natural oscillation (no external forces applied).
2. Periodic external force present.
3. Random stochastic force present.

This report mainly focuses on the results of the modelling.

Van der pol

The Van der Pol oscillator (Kanamaru 2007) is described by the following nonlinear differential equation:

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0$$

The variable x is a function of time t, and the constant $\varepsilon \geq 0$ is a *control parameter*. Its value is crucial for the nonlinear behaviour of the oscillation, as it controls two major characterisations of the system:

1. *Damping* – how strongly the oscillation is pulled to the origin when x becomes large.
2. *Feedback* – how strongly the oscillation is pushed through the origin when x becomes small.

The Van der Pol oscillator was modelled for a number of values of ε, and the results are shown in Fig. 1 for small ε, and Fig. 2 for large ε. Both have initial conditions $(x, \dot{x}) = (x, v) = (0,0.5)$.

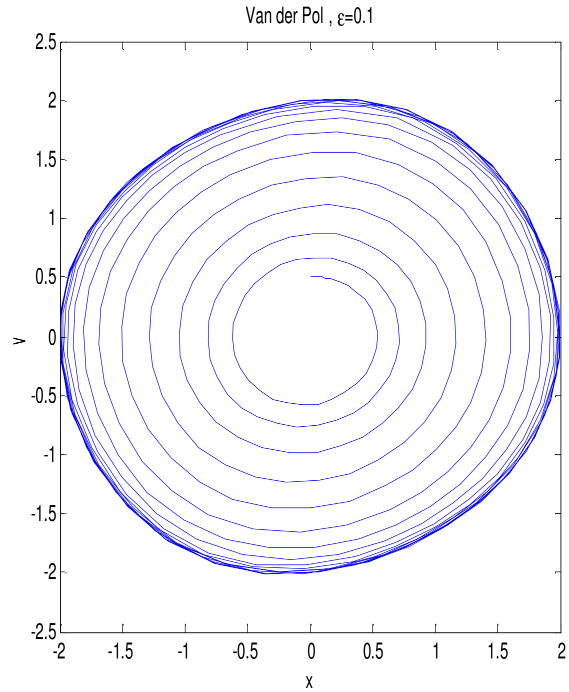

Figure 1: x-v plot showing limit cycle for small ε

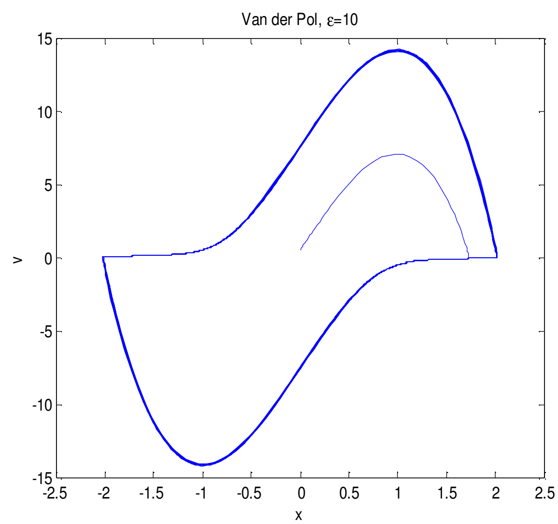

Figure 2: x-v plot showing limit cycle for large ε

Both oscillations reach a *limit cycle*, which is an attractive equilibrium for the oscillator. This limit cycle is what makes the system *self-organising*, as it will be attracted to the limit cycle for all initial conditions other than (0,0). The larger ε is, the more distorted the limit cycle gets from a circle, as the damping and feedback forces increase. The system can be thought of as having more intrinsic energy.

Applying periodic forcing

An external force can be simulated as acting on the system by adding a new periodic force to the equation (Ulrich Parlitz 1987):

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = F cos\left(\frac{2\pi t}{T_{in}}\right)$$

This force is defined by F, its amplitude, and T_{in}, its period. Figs. 3 and 4 show the resulting limit cycles when $F = 1.2$ and $T_{in} = 10$, when = 6 and 7.3 respectively.

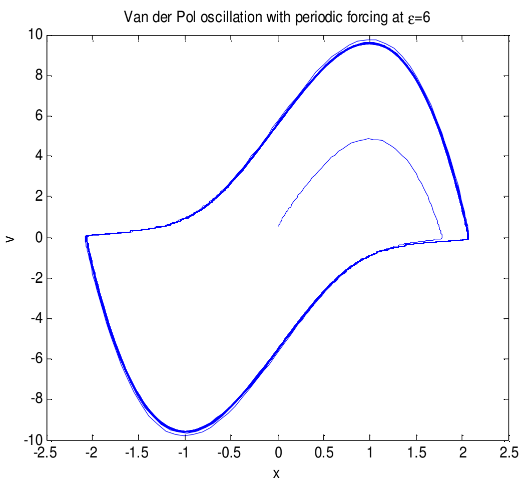

Figure 3: Limit cycle for periodic result

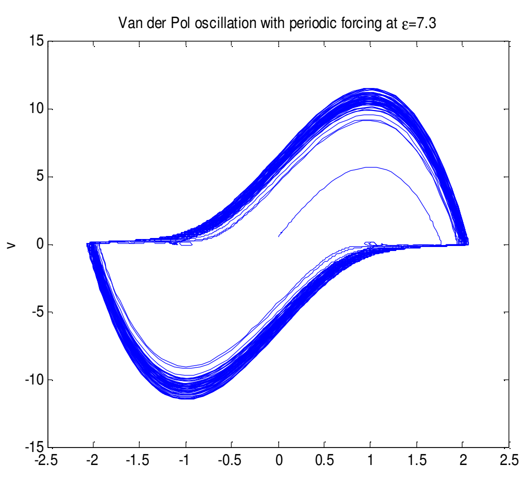

Figure 4: Limit cycle for chaotic result

Despite these values of being close, they reveal very different outcomes, as Fig. 3 shows a limit cycle similar to the natural oscillations from before, and Fig. 4 has a more broadened limit cycle, with some distinctly chaotic looking loops at small v.

These results become clear when the period T_{out} of the whole system is examined in more detail for more values of , as in Fig. 5. These values of T_{out} were calculated from the Fourier transform of the oscillations.

Where the graph shows a plateau, the results are periodic, i.e. have a limit cycle similar in appearance to Fig 3. The sections of the graph where it is climbing are known as *windows of chaos*, and here the limit cycle is broadened as in Fig. 4.

The graph also illustrates *period doubling,* a phenomenon well known in chaos theory (Ott 1993). The self-organisation of the system results in $T_{out} = nT_{in}$ where n is an integer; in these cases $n = 2{,}3$ but as increases it will take on larger integers.

There are also *quasiperiods* present where $T_{out} = 16.67$ and 23.33. This is when the period is multiplied by a fraction rather than an integer:

$$T_{out} = \frac{n}{m} T_{in}$$

This is the general result, and encompasses normal period doubling where $m = 1$.

Another property of the graph is that as F increases, the windows of chaos decrease in size, so an external force with a larger amplitude is more likely to give a periodic result.

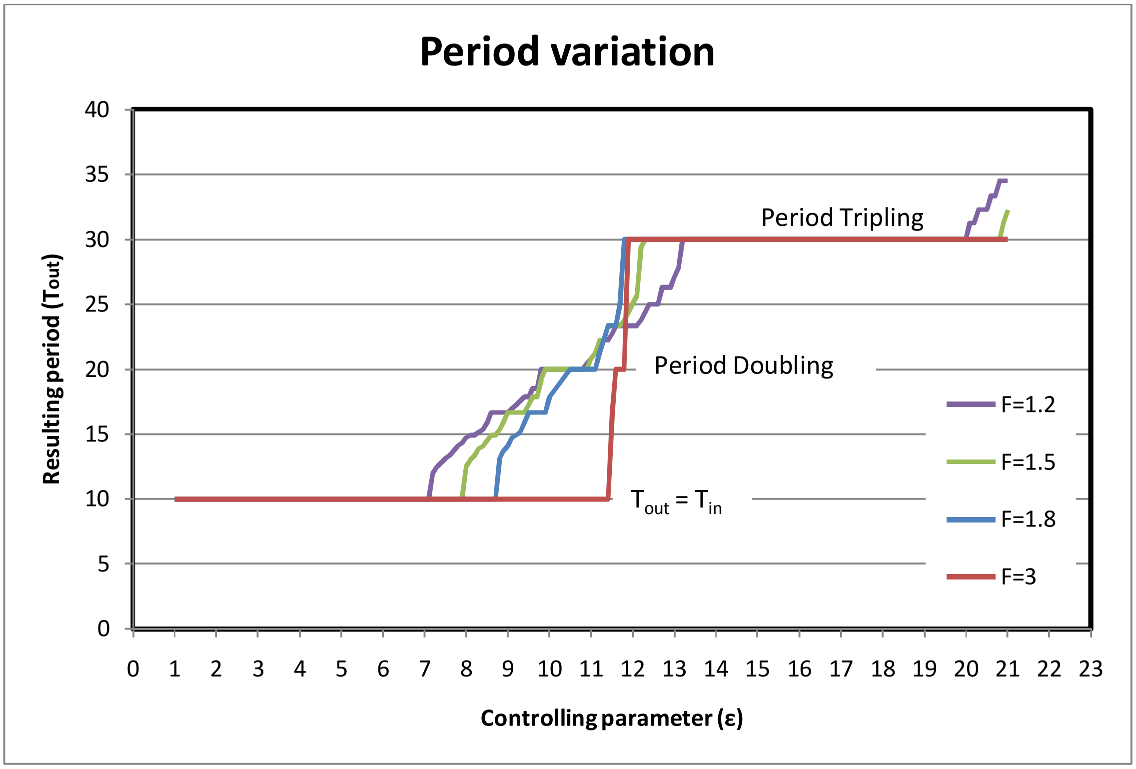

Figure 5: Graph showing period output of the Van der Pol oscillator under periodic external force

Random forcing

It was also tested to see how the system would react to random force being applied. This is a useful way of modelling unknown stochastic forces or noise. It was modelled by replacing T_{in} with a random stream of independent uniformly distributed numbers from 0 to 20. The average would be 10, so as to allow comparison to the periodic external force. (A. Zakharova 2010), (Leung 1995), (Shinji Doi 2003)

It was found that one very important factor in the outcome was how quickly the random stream changed, i.e. the *correlation time*. The smaller the correlation time (the quicker the changeover), the less the system was affected, regarding its amplitude, period and shape. This is illustrated in Fig. 6.

Another major factor is the magnitude of F. However, the value of ε appears to make little to no difference to what kind of effect the random forcing has on the system, unlike with periodic forcing.

To further investigate the results, *probability density functions* (PDFs) and Fourier transforms were carried out. A PDF shows how often a certain value occurs, so illustrates the general distribution of the points. Fig. 7 shows a PDF of a randomly forced result, compared with PDFs of the natural oscillation and periodic external force. It appears that the randomly forced result has more in common with the unforced result, since the periodically forced result has a chaotic PDF due to the chosen value of □. If □ is chosen so there is a periodic result from periodic forcing, then all graphs look very similar in shape and it is more difficult to make a comparison.

The Fourier transforms were carried out to discover if T_{out} was altered by random forcing. It was generally found that when $F = 1.2$ the period was not changed, but increasing F often resulted in a change of T_{out}, either up or down depending on the random stream.

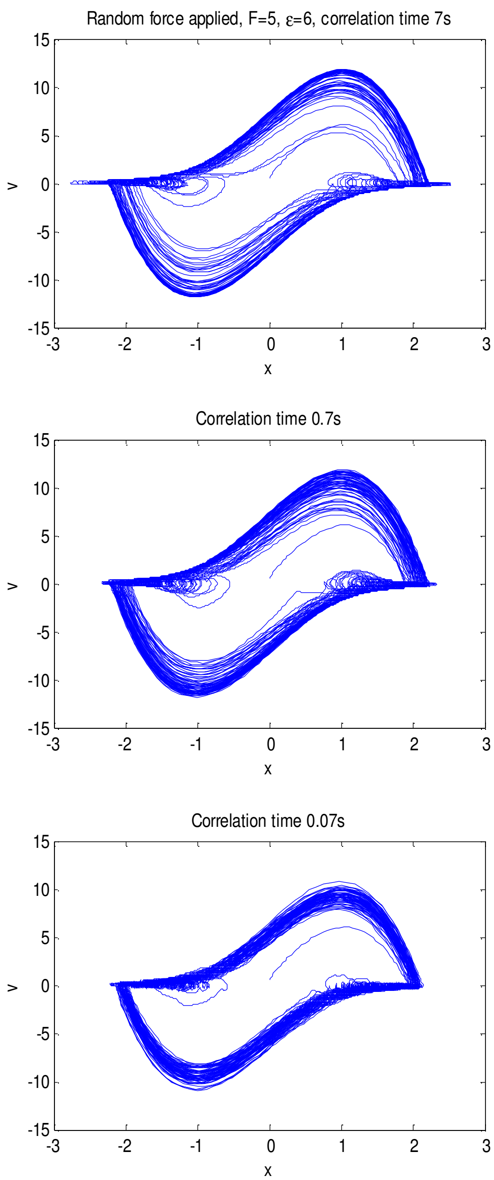

Figure 6: Changing correlation time with external random forcing

Summarising the results from PDFs and Fourier transforms: external random forcing gives a very different result to external periodic forcing. Removing the predictability of the external force results in the system behaving more like the original oscillation, albeit with a broader limit cycle. This then becomes broader if the correlation time or F is increased.

PDFs also show themselves to be a good method of displaying information about a nonlinear system, as they can show indications of chaos and the size of the system.

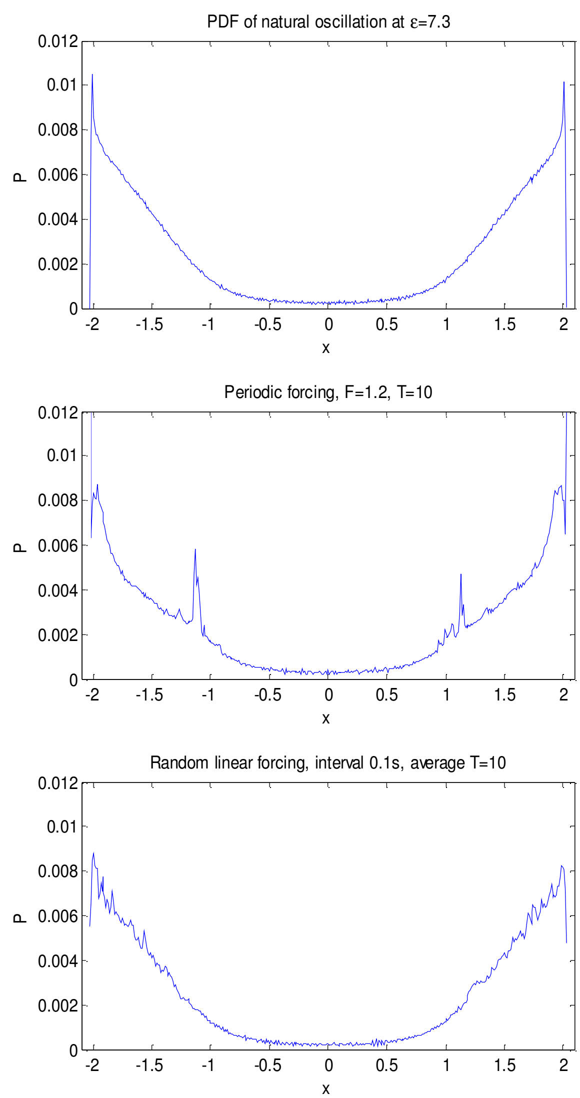

Figure 7: Probability density functions of x under the different external conditions

Conclusions and Further Work

There are not many conclusions to be reached at this stage, as this first half of the project was just to get used to the nonlinear dynamical systems in self-organised criticality.

However, these limited conclusions could be reached:

1. The Van der Pol oscillator will self-organise into a limit cycle if there is no external force applied, and the shape of the limit cycle depends on the value of □.
2. If an external oscillating force with period T_{in} is applied, the oscillator will either remain periodic, changing its period to a multiple of T_{in}, or turn chaotic.
3. If a random force is applied, the period will remain similar if the amplitude of the force is small, or change unpredictably (within a certain range) for larger amplitudes.
4. In all situations, the system remains bounded within a certain range, relating to the values of F and □.

In the second half of the project, these results will be generalised and linked to a generalised concept of entropy. This is hoped to be achieved by linking the PDFs with Shannon entropy (Beck 2009).

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